AN EXPLICIT COMPUTATION OF CHROMATIC GRAPH HOMOLOGY

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The computations in this note will probably only be useful to me, but the only other similar computation I can find on the internet is (here) for the Khovanov homology of the trefoil.

We consider the chromatic complex as defined in [HGR05] over the algebra $A_2 = \mathbb{Z}[x]/(x^2)$. In particular, we consider the complex for the triangle graph which we denote P_3 following the notation in [HGR05] (P stands for polygon). This is far from a novel calculation but here we explicitly compute the matrices that make up the per-edge maps. We also explicitly compute the differentials as matrices by taking the direct sum of the per-edge map matrices. In other words, we stack them on top of each other.

The complex for the triangle graph is pictured in Figure 2. We label the vertices of the graph as in Figure 1. We consider an ordering on the components induced by the smallest labeled vertex in the component. This tells us that when going from the state with no edges to the state with the edge 13, that the component containing the vertices 1 and 3 comes first. That means given $v_1 \otimes v_2 \otimes v_3 \in C^0(G)$, the portion of the image of the map that ends in the 13 state will be $v_1v_3 \otimes v_2$ as opposed to $v_2 \otimes v_1v_3$ since we consider the 13 component to come first.

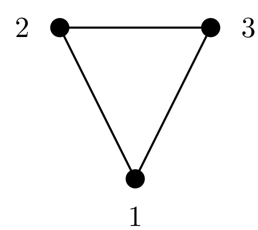

FIGURE 1. Labeling of the triangle graph P_3

We define f_1 as below

	$1\otimes 1\otimes 1$	$1\otimes 1\otimes x$	$1\otimes x\otimes 1$	$x\otimes 1\otimes 1$	$1\otimes x\otimes x$	$x\otimes 1\otimes x$	$x\otimes x\otimes 1$	$x\otimes x\otimes x$
$1\otimes 1$	1	0	0	0	0	0	0	0
$1\otimes x$	0	1	0	0	0	0	0	0
$x\otimes 1$	0	0	1	1	0	0	0	0
$x\otimes x$	0	0	0	0	1	1	0	0

We define f_2 as below

	$1\otimes 1\otimes 1$	$1\otimes 1\otimes x$	$1\otimes x\otimes 1$	$x\otimes 1\otimes 1$	$1\otimes x\otimes x$	$x\otimes 1\otimes x$	$x\otimes x\otimes 1$	$x\otimes x\otimes x$
$1\otimes 1$	1	0	0	0	0	0	0	0
$1\otimes x$	0	1	1	0	0	0	0	0
$x\otimes 1$	0	0	0	1	0	0	0	0
$x\otimes x$	0	0	0	0	0	1	1	0

Date: January 8, 2021.

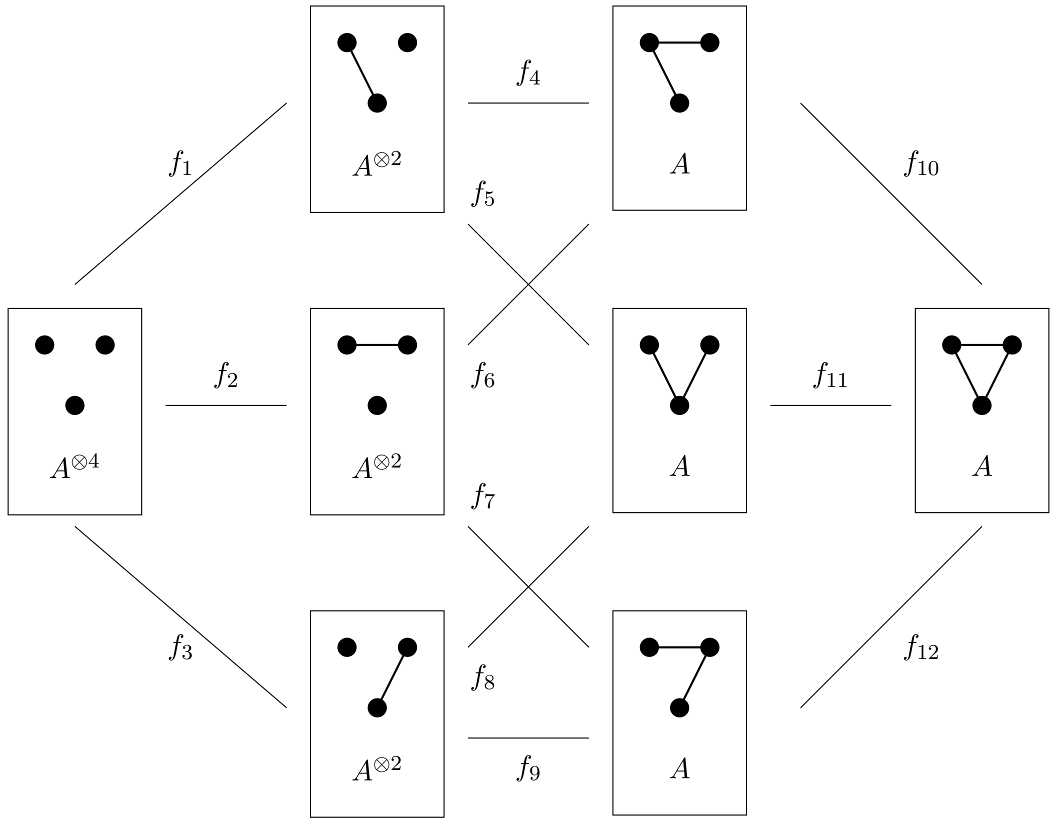

FIGURE 2. Cubical complex for the triangle graph

We define f_3 as below

	$1\otimes 1\otimes 1$	$1\otimes 1\otimes x$	$1\otimes x\otimes 1$	$x\otimes 1\otimes 1$	$1\otimes x\otimes x$	$x\otimes 1\otimes x$	$x\otimes x\otimes 1$	$x\otimes x\otimes x$
$1\otimes 1$	0	0	0	0	0	0	0	0
$1\otimes x$	0	0	1	0	0	0	0	0
$x\otimes 1$	0	1	0	1	0	0	0	0
$x\otimes x$	0	0	0	0	1	0	1	0

Stacking f_1 on top of f_2 on top of f_3 yields d^0. That is,

$$d^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

We next compute $f_4, \ldots, f_9$ as below. Recall that we sprinkle in minus signs based on the ordering of the edge being added to the current state. The sign taken is -1 to the power of the number of edges preceding (under the induced lexicographic order of the edges) the edge being added.

$$f_4 = \begin{array}{c} \\ 1 \\ x \end{array} \begin{array}{c} \begin{array}{cccc} 1\otimes 1 & 1\otimes x & x\otimes 1 & x\otimes x \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix} \end{array}$$

$$f_5 = \begin{array}{c} \\ 1 \\ x \end{array} \begin{array}{c} \begin{array}{cccc} 1\otimes 1 & 1\otimes x & x\otimes 1 & x\otimes x \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix} \end{array}$$

$$f_6 = \begin{array}{c} \\ 1 \\ x \end{array} \begin{array}{c} \begin{array}{cccc} 1\otimes 1 & 1\otimes x & x\otimes 1 & x\otimes x \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

$$f_7 = \begin{array}{c} \\ 1 \\ x \end{array} \begin{array}{c} \begin{array}{cccc} 1\otimes 1 & 1\otimes x & x\otimes 1 & x\otimes x \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

$$f_8 = \begin{array}{c} \\ 1 \\ x \end{array} \begin{array}{c} \begin{array}{cccc} 1\otimes 1 & 1\otimes x & x\otimes 1 & x\otimes x \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

$$f_9 = \begin{array}{c} \\ 1 \\ x \end{array} \begin{array}{c} \begin{array}{cccc} 1\otimes 1 & 1\otimes x & x\otimes 1 & x\otimes x \end{array} \\ \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{pmatrix} \end{array}.$$

We can now combine the maps as follows to form d^1

$$d^1 = \begin{bmatrix} f_4 & f_6 & 0 \\ f_5 & 0 & f_8 \\ 0 & f_7 & f_9 \end{bmatrix}.$$

To see why this is the case we take the direct sum of maps leaving the same state causing us to stack those maps in the matrix. We place the maps from the next states moving down a column of the cube as in Figure 2 to the right of the maps from the states above them. This way when we multiply d^1 by a vector representing entries from a specific state we combine the outputs from all maps leaving that state and ignore the others.

More explicitly, we have d^1 :

$$d^1 = \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & -1 & 0 \end{bmatrix}.$$

We've basically got the hang of it at this point and compute $f_10, \dots, f_{12}$ below:

$$f_{10} = \begin{array}{c} \\ 1 \\ x \end{array} \begin{array}{c} \begin{array}{cc} 1 & x \end{array} \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}$$

$$f_{11} = f_{12} = \begin{array}{c} \\ 1 \\ x \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & x \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array}.$$

We combine these maps as follows to get d^2:

$$\begin{aligned} d^2 &= \begin{bmatrix} f_10 & f_11 & f_12 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}. \end{aligned}$$

If we care only about the homological degree of homology then it's enough to just consider the Smith normal form of these matrices and compute homology from that.

Theorem 1 ([Zom05])**.** *Let R be a PID and let M and N be matrices with*

$$R^m \xrightarrow{M} R^n \xrightarrow{N} \dots.$$

Let the diagonal elements of $\mathrm{SNF}(M)$ *be $m_1, \dots, m_r$ and the diagonal elements of* $\mathrm{SNF}(N)$ *be $n_1, \dots, n_s$ then*

$$\ker(N)/\operatorname{im}(M) \cong R^{n-r-s} \oplus \bigoplus_{i=1}^{r} R/(m_i R).$$

Using Mathematica (or really whatever software you like to compute the Smith normal form) we have

$$\mathrm{SNF}(d^0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{SNF}(d^1) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and finally

$$\mathrm{SNF}(d^2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Applying Theorem 1 tells us $H^0(P_3) \cong mathbbZ$, $H^1(P_3) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ and $H^2(P_3) = 0$ by computation. We also have $H^i(P_3) = 0$ for $i > 2$ since all higher chain complexes are 0 for a graph on 3 edges.

References

[HGR05] Laure Helme-Guizon and Yongwu Rong. A categorification for the chromatic polynomial. *Algebraic & Geometric Topology*, 5(4):1365–1388, 2005.

[Zom05] Afra J Zomorodian. *Topology for computing*, volume 16. Cambridge university press, 2005.