THE MÖBIUS BAND AS A FIBER BUNDLE

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This is just a quick note intended to introduce the definition of fiber bundles and explain why the Möbius band is an example of one.

Definition 1. *A* ***fiber bundle*** *structure on a space E with fiber F consists of a projection map $p : E \to B$ such that for each $b \in B$ there exists a neighborhood U_b such that there exists a homeomorphism $h : p^{-1}(U_b) \to U_b \times F$ which makes the following diagram commute*

$$\begin{array}{ccc} p^{-1}(U_b) & \xrightarrow{h} & U_b \times F \\ & \searrow^{p} & \downarrow{\pi} \\ & & U_b \end{array}$$

where π is the projection onto the first factor [Hat00].

A Möbius band is one of the simplest examples of a nontrivial fiber bundle. It is a fiber bundle with base space S^1 and fiber a closed interval. By the definition of fiber bundle, for each point in S^1 we will need a neighborhood for which we can find a homeomorphism h as described previously. Splitting S^1 into any two open arcs which cover the circle will do. We choose U_1 to be S^1 without the point at 0 radians and we choose U_2 to be S^1 without the point at π radians as pictured in Figure 1. We can picture the Möbius band through its fundamental polygon, a rectangle with the

FIGURE 1. Our chosen neighborhoods U_1 and U_2 that cover S^1

left and right hand sides identified with a twist. That is, the left hand side of the rectangle is oriented up and the right hand side is oriented down. This is exactly what happens when we build one out of a rectangular strip of paper, we glue together the two edges but we have to put a twist into the paper. With this image of the band in mind, we will now picture the inverse images of U_1 and U_2 within the band. First consider $p^{-1}(U_1)$ as in Figure 2. We see $p^{-1}(U_1)$ above the base space with U_1 pictured within the base space. In $p^{-1}(U_1)$ the left and right edges of the Möbius band are not colored blue since these are precisely the fiber of $S^1 \setminus U_1$. This is pictured more clearly on the right hand side of the figure where we see the space is homeomorphic to a closed interval crossed with an open interval. This in turn is homeomorphic to $U_1 \times [0, 1]$ as required for this to be a fiber bundle structure. It is also obvious from the picture that the diagram commutes since points are mapped directly across to their counterparts as pictured. The situation for U_2 is only slightly more complicated. Consider Figure 3 and this time the left and right hand sides are

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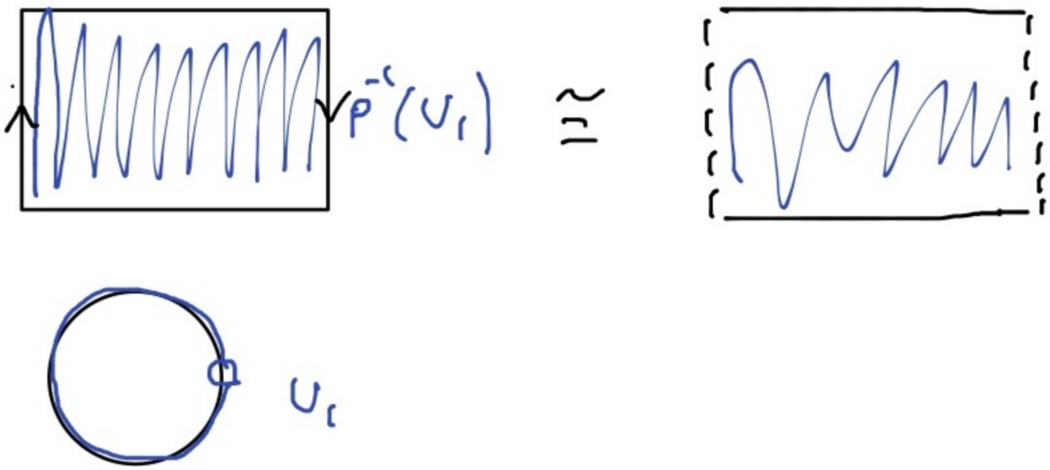

FIGURE 2. $p^{-1}(U_1)$ in the Möbius band

colored, but this time we are missing a line vertically through the middle of the rectangle. This is the fiber of the point $S^1 \setminus U_2$. Again we see that we have a closed interval cross an open interval as required. Our homeomorphism "untwists" the band so that the right hand side is now directed upwards. This maps the points on the left hand side of the dotted line to their counterparts but on the right side we now map points to their reflection across the horizontal axis. In either case, the maps commute as pictured with an example point, since flipping across the horizontal access remains within the same fiber. Therefore, the Möbius band is a fiber bundle over S^1 with fiber an interval.

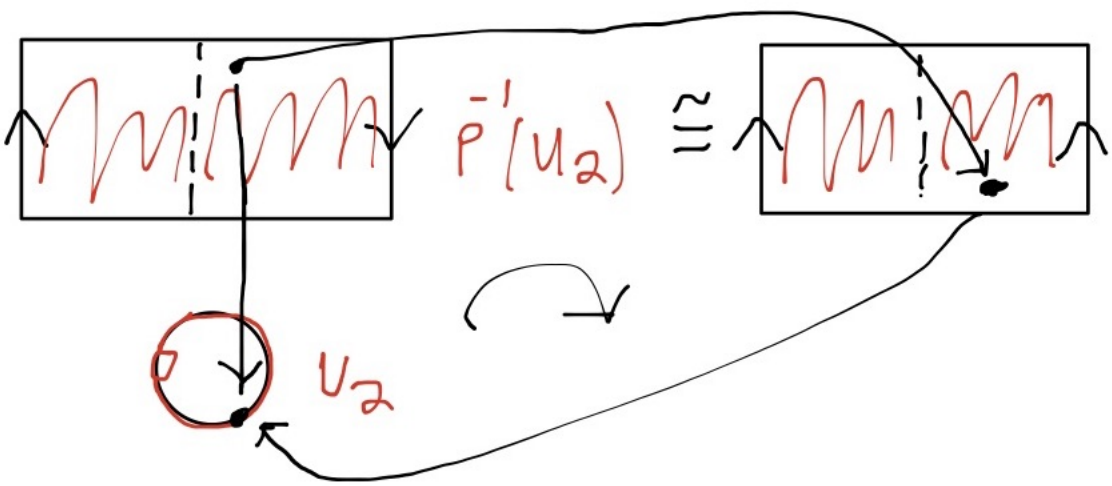

FIGURE 3. $p^{-1}(U_2)$ in the Möbius band

References

[Hat00] Allen Hatcher. *Algebraic topology.* Cambridge Univ. Press, Cambridge, 2000.